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ON THE NITSCHE CONJECTURE FOR HARMONIC MAPPINGS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

BY

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ABSTRACT

We give the new inequality related to the J. C. C. Nitsche conjecture (see [6]). Moreover, we consider the two- and three-dimensional case. Let $A(r,1) = \{z : r < |z| < 1\}$. Nitsche's conjecture states that if there exists a univalent harmonic mapping from an annulus $A(r,1)$ to an annulus $A(s,1)$, then s is at most $2r/(r^2+1)$.

Lyzzaik's result states that $s < t$ where t is the length of the Grötzsch's ring domain associated with $A(r,1)$ (see [5]). Weitsman's result states that $s \leq 1/(1+1/2(r\log r)^2)$ (see [8]).

Our result for two-dimensional space states that $s \leq 1/(1+1/2\log^2 r)$ which improves Weitsman's bound for all r, and Lyzzaik's bound for r close to 1. For three-dimensional space the result states that $s \leq 1/(r - \log r)$.

1. Introduction and auxiliary results

Let $\mathbb{R}^n$ be the real n-space with the norm $||x|| = (\sum_{i=1}^n x_i^2)^{1/2}$ and basis $e_1 = (1,0,\dots,0)^T, \dots, e_n = (0,\dots,0,1)^T$. If $x_1,\dots,x_n \in \mathbb{R}^n$ are vectors $x_i = \sum_{j=1}^n x_{ij}e_j$, $i = 1,\dots,n$ then their vector product $x_1 \times \cdots \times x_{n-1}$ is defined as the vector

$$x = x_1 \times \cdots \times x_{n-1} = \begin{vmatrix} e_1 & e_2 & \cdots & e_n \\ x_{11} & x_{12} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n} \end{vmatrix},$$

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the determinant being developed w.r.t. the first line. The linear operator A: $\mathbb{R}^n \to \mathbb{R}^n$ is identified with matrices $A = [a_{ij}]_{i,j=1,\ldots,n}$. Two norms of A are considered:

$$||A|| = \sup\{||Ax|| : ||x|| = 1\} \quad \text{and} \quad ||A||_2 = \left(\sum_{i=1}^{n} a_{ij}^2\right)^{1/2}.$$

Let $x = \sum_{i=1}^n x_i e_i$. Then $Ax = \sum_{i=1}^n A(x_i e_i) = \sum_{i=1}^n x_i A e_i$. It follows that

$$\begin{aligned} ||Ax|| &\leq \sum_{i=1}^{n} |x_i| ||Ae_i|| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} ||Ae_i||^2\right)^{1/2} \\ &= ||x|| \cdot \left(\sum_{i=1}^{n} a_{ij}^2\right)^{1/2} = ||x|| \cdot ||A||_2. \end{aligned}$$

It follows at once that

$$||A|| \leq ||A||_2. \tag{1.1}$$

A twice differentiable mapping u defined between the domains Ω and Ω' of the n-dimensional Euclidean space is called **harmonic** if it satisfies the differential equation $\Delta u = D_{11}u + \cdots + D_{nn}u = 0$. The set $R = A(r_1, r_2) = \{r_1 < ||x|| < r_2\}$ $(0 \leq r_1 < r_2 \leq \infty)$ we will call an annulus. We define the modulus $m(R)$ of the annulus R by the formula

$$m(R) = m(A(r_1, r_2)) = \frac{1}{\omega_{n-1}} \log^{n-1}\left(\frac{r_2}{r_1}\right),$$

where ω_{n-1} is the $(n-1)$-dimensional Lebesgue measure of the unit sphere S^{n-1}.

With each doubly connected domain R in the complex plane there is associated a unique number $\alpha > 1$ such that R can be mapped one-to-one and conformally onto an annulus whose outer and inner radii are in the ratio $\alpha : 1$.

Let Γ be the family Γ' of Jordan rectrifiable arcs on R which connect the components of connectivity of the set R^c or the family Γ'' of Jordan rectifiable curves that separate the components of connectivity of the set R^c if $n = 2$. Let p be a positive integrable function defined on R such that the integral

$$A(p) = \int_R p^n$$

is a positive real number. Then we say that p is an admissible metric. The family of the admissible metrics we will denote by Π. The extremal length of the family Γ, with respect to metric p, is defined by

$$L_p = \inf_{\gamma \in \Gamma} \int_\gamma p ds.$$

PROPOSITION 1.1 (see [1] and [2]): *If $R = A(r_1, r_2)$ is an annulus and if $m(R)$ is the modulus of $A(r_1, r_2)$, then*

$$(1.2) \qquad m(R) = \begin{cases} \inf_{p \in \Pi} \frac{A(p)}{L_p^n} & \text{if } n = 2 \text{ and } \Gamma = \Gamma'', \\ \sup_{p \in \Pi} \frac{L_p^n}{A(p)} & \text{if } \Gamma = \Gamma'. \end{cases}$$

The relation (1.2) defines the modulus of an arbitrary doubly connected domain R in $\mathbb{R}^n$.

LEMMA 1.2: *Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator such that $A = [a_{ij}]_{i,j=1,\dots,n}$. Then*

$$(1.3) \qquad ||Ax_1 \times \cdots \times Ax_{n-1}|| \leq ||A||_2^{n-1} ||x_1 \times \cdots \times x_{n-1}||.$$

Proof: Let

$$x_i = \sum_{j=1}^n x_{ij} e_j, \quad \text{and} \quad y_i = Ax_i = \sum_{j=1}^n y_{ij} e_i \quad \text{for } i \in \{1, \dots, n-1\}.$$

Let $\tilde{A} = [\tilde{a}_{ij}]$ be the matrix defined by

$$\tilde{a}_{ij} = (-1)^{i+j} \det([a_{lk}]_{l=1,\dots,i-1,i+1,\dots n}^{k=1,\dots,j-1,j+1,\dots n}).$$

Then

$$\begin{aligned} &Ax_1 \times \cdots \times Ax_{n-1} \\ &= \sum_{i=1}^n \sum_{j \in S_{n,i}} \varepsilon_j x_{1,j_1} \cdot \ldots \cdot x_{n-1,j_{n-1}} Ae_1 \times \cdots \times Ae_{i-1} \\ &\quad \times Ae_{i+1} \times \cdots \times Ae_n \\ &= \sum_{i=1}^n \sum_{j \in S_{n,i}} \varepsilon_j x_{1,j_1} \cdot \ldots \cdot x_{n-1,j_{n-1}} (\tilde{a}_{1i}, \dots, \tilde{a}_{ni}) \\ &= \sum_{i=1}^n \sum_{j \in S_{n,i}} \varepsilon_j x_{1,j_1} \cdot \ldots \cdot x_{n-1,j_{n-1}} (-1)^{i+1} \tilde{A} e_i \\ &= \sum_{i=1}^n \sum_{j \in S_{n,i}} \varepsilon_j x_{1,j_1} \cdot \ldots \cdot x_{n-1,j_{n-1}} \tilde{A} e_1 \times \cdots \times e_{i-1} \times e_{i+1} \times \cdots \times e_n \\ &= \tilde{A} x_1 \times \cdots \times x_{n-1}, \end{aligned}$$

where $S_{n,i}$ is the set of the permutations of the set $\{1,\dots,n\}\setminus\{i\}$, and ε_j is the sign of the permutation j. Now, using the inequality (1.1), we obtain

$$\begin{aligned}||Ax_1\times\cdots\times Ax_{n-1}|| &\le ||\tilde{A}||||x_1\times\cdots\times x_{n-1}|| \le ||\tilde{A}||_2||x_1\times\cdots\times x_{n-1}||\\ &= \Big(\sum_{i,j=1}^n \tilde{a}_{ij}^2\Big)^{1/2}||x_1\times\cdots\times x_{n-1}||.\end{aligned}$$

Next, we easily obtain that

$$\Big(\sum_{i,j=1}^n \tilde{a}_{ij}^2\Big)^{1/2} \le \sum_{i=1}^n\Big(\sum_{j=1}^n \tilde{a}_{ij}^2\Big)^{1/2} = \sum_{i=1}^n ||Ae_1\times\cdots\times Ae_{i-1}\times Ae_{i+1}\times\cdots\times Ae_n||.$$

It follows at once that

$$\Big(\sum_{i,j=1}^n \tilde{a}_{ij}^2\Big)^{1/2} \le (||Ae_1||^2+\cdots+||Ae_n||^2)^{(n-1)/2} = ||A||_2^{n-1}.$$

This completes the proof of the lemma. ∎

If $S=(S_1,S_2,\dots,S_n)\colon \mathbb{R}^n\to\mathbb{R}^n$ and if ∇ denotes the gradient of the real function, then we have $S'=(\nabla S_1,\dots,\nabla S_n)$.

LEMMA 1.3: *Let u be a harmonic mapping. Let $u=\rho\cdot S$, where $\rho=||u||$. Then*

$$\Delta\rho = \rho||S'||_2^2. \tag{1.4}$$

Proof: Let $u=(u_1,u_2,\dots,u_n)$ (here u_i are real harmonic), and let $S=(S_1,S_2,\dots,S_n)$. Since $u_i=\rho S_i$, $i\in\{1,\dots,n\}$ and consequently

$$\rho=\sum_{i=1}^n S_iu_i,$$

we obtain

$$0=\Delta u_i = \Delta\rho S_i + \Delta S_i\rho + 2\nabla\rho\nabla S_i,\quad i\in\{1,\dots,n\} \tag{1.5}$$

and

$$\Delta\rho = \sum_{i=1}^n u_i\Delta S_i + 2\sum_{i=1}^n \nabla S_i\cdot\nabla u_i. \tag{1.6}$$

From (1.5) we obtain

$$\Delta\rho = \Delta\rho||S||^2 = \sum_{i=1}^n S_i\cdot\Delta\rho S_i = -\rho\sum_{i=1}^n S_i\Delta S_i - 2\sum_{i=1}^n S_i\nabla\rho\cdot\nabla S_i. \tag{1.7}$$

Adding (1.6) and (1.7) we obtain

$$\Delta\rho = \sum_{i=1}^{n}(\nabla S_i \cdot \nabla u_i - S_i \nabla\rho \cdot \nabla S_i) = \rho \sum_{i=1}^{n} \|\nabla S_i\|^2.$$

This completes the proof. ∎

Let f be a function between A and B. By $N(y,f)$ we denote the cardinal number of $f^{-1}(y)$ if the last set is finite and we set $N(y,f) = +\infty$ in the other case. The function $y \to N(y,f)$ is defined on B. If f is surjective then $N(y,f) \geq 1$ for every $y \in B$. The following proposition holds.

PROPOSITION 1.4 ([7]): *Let U be an open subset of $\mathbb{R}^n$ and let $f\colon U \to \mathbb{R}^n$ be C^1 mapping. Then the function $y \to N(y,f)$ is measurable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} N(y,f)\mathrm{d}y = \int_U |J(x,f)|\mathrm{d}x, \tag{1.8}$$

where $J(x,f)$ is the Jacobian of f.

Further, let h be a C^1 surjection from an $(n-1)$-dimensional rectangle K^{n-1} onto the unit sphere S^{n-1}. Let the function f be defined in the n-dimensional rectangle $K^n = [0,1] \times K^{n-1}$ by $f(r,u) = rh(u)$. Thus f is a C^1 surjection from K^n onto the unit ball B^n. It is easy to obtain the formula $J(x,f) = r^{n-1}D_h(u)$, where $x = (r,u) \in K^n$, and D_h denotes the norm of the vector product

$$D_h = \left\| \frac{\partial h}{\partial x_1} \times \cdots \times \frac{\partial h}{\partial x_{n-1}} \right\|.$$

According to Proposition 1.4 it follows that

$$\begin{aligned}\frac{1}{n}\omega_{n-1} &= \mu(B^n) = \int_{B^n} \mathrm{d}y \leq \int_{B^n} N(y,f)\mathrm{d}y \\ &= \int_{K^n} |J(x,f)|\mathrm{d}x = \int_0^1 r^{n-1}\mathrm{d}r \int_{K^{n-1}} D_h(u)du = \frac{1}{n}\int_{K^{n-1}} D_h(u)du.\end{aligned}$$

Consequently we have

$$\int_{K^{n-1}} D_h(u)du \geq \omega_{n-1}. \tag{1.9}$$

PROPOSITION 1.5: *Let u be a C^1 surjection between the spherical rings $A(r_1,r_2)$ and $A(s_1,s_2)$, and let $S = u/\|u\|$. Let P^{n-1} be a closed $(n-1)$-dimensional hypersurface that separates the components of the set $A^C(r_1,r_2)$. Then*

$$\int_{P^{n-1}} \|S'\|_2^{n-1} dP \geq \omega_{n-1} \tag{1.10}$$

and

$$\int_{A(r_1,r_2)} ||S'||_2^{n-1} dA \geq (r_2 - r_1)\omega_{n-1}, \tag{1.11}$$

where ω_{n-1} denotes the measure of S^{n-1}.

Proof: Let K^{n-1} be an $(n-1)$-dimensional rectangle and let $g\colon K^{n-1} \to P^{n-1}$ be a parametrization of P^{n-1}. Then the function $S \circ g$ is a differentiable surjection from K^{n-1} onto the unit sphere S^{n-1}. Then by (1.9) we have

$$\int_{K^{n-1}} D_{S\circ g} dK \geq \omega_{n-1}.$$

According to Lemma 1.2 we obtain

$$D_{S\circ g}(x) = \left\| S'(g(x))\frac{\partial g(x)}{\partial x_1} \times \cdots \times S'(g(x))\frac{\partial g(x)}{\partial x_{n-1}} \right\| \leq ||S'(g(x))||_2^{n-1} D_g(x).$$

Hence we obtain

$$\omega_{n-1} \leq \int_{K^{n-1}} ||S'(g(x))||_2^{n-1} D_g(x) dK(x) = \int_{P^{n-1}} ||S'(\zeta)||_2^{n-1} d\sigma(\zeta).$$

Thus we have proved (1.10). It follows that

$$\int_{A(r_1,r_2)} ||S'||_2^{n-1} dA = \int_{r_1}^{r_2} \left(\int_{S^{n-1}(0,t)} ||S'||_2^{n-1} \mathrm{d}S \right) \mathrm{d}t \geq (r_2 - r_1)\omega_{n-1}.$$

The proof of the theorem has been completed. ∎

Proposition 1.6: *Let $u = \rho e^{i\Theta}$ be a C^1 surjection between the rings $A(r_1, r_2)$ and $A(s_1, s_2)$ of the complex space. Then*

$$\int_{r_1 \leq |z| \leq r_2} |\nabla\Theta|^2 \mathrm{d}x\mathrm{d}y \geq 2\pi \log\frac{r_2}{r_1}. \tag{1.12}$$

Proof: Because $|\nabla\Theta|$ is an admissible metric, according to Proposition 1.1, we have

$$\frac{\int_R |\nabla\Theta|^2 \mathrm{d}x\mathrm{d}y}{\inf_{\gamma\in\Gamma}(\int_\gamma |\nabla\Theta||\mathrm{d}z|)^2} \geq m(R) = \inf_\rho \frac{A(\rho)}{L^2(\rho)} = \frac{1}{2\pi}\log\frac{r_2}{r_1}, \tag{1.13}$$

where $R = A(r_1, r_2)$ and Γ is the subset of curves which separates the components of connectivity of R^C, and $m(R)$ is the modulus of R.

Next, if γ is Jordan curve, then the function $\Theta\colon \gamma\backslash\gamma(0) \to (0, 2\pi)$ is surjective. According to (1.10) we obtain

$$\int_\gamma |\nabla\Theta||\mathrm{d}z| \geq 2\pi. \tag{1.14}$$

Observe that, in this case, $S = e^{i\Theta}$ and $||S'||_2 = |\nabla\Theta|$. (1.13) and (1.14) yield (1.12). ∎

2. The main result

THEOREM 2.1: *Let there be a harmonic surjection between the annular regions $A(r_1, r_2)$ and $A(s_1, s_2)$ of $\mathbb{R}^n$, $n = 2, 3$, satisfying the conditions $||x|| \to r_i \Rightarrow ||u(x)|| \to s_i$, $i = 1, 2$. Then*

$$\frac{s_2}{s_1} \geq \begin{cases} 1 + \frac{1}{2}\log^2\frac{r_2}{r_1} & \text{if } n = 2, \\ \log\frac{r_2}{r_1} + \frac{r_1}{r_2} & \text{if } n = 3. \end{cases} \tag{2.1}$$

Note that if u is a harmonic homeomorphism, then it satisfies the conditions of the theorem.

Proof: Let u be a harmonic surjection between the corresponding annular regions. For $n > n_0 > \max\{2, 1/(r_2 - r_1)\}$ let

$$s_n = \sup(\{||y|| : y \in A(s_1, s_2) \wedge y \notin u(A(r_1 + 1/n, r_2))\} \cup \{s_1\}).$$

If $y \in A(s_1, s_2) \wedge y \notin u(A(r_1 + 1/n, r_2))$ then $||y|| \leq s_n$, hence $y \notin A(s_n, s_2)$. Consequently, $A(s_n, s_2) \subset B_n = u(A(r_1 + 1/n, r_2))$. The sequence s_n is decreasing. Hence it is a convergent sequence. Consequently, only one of the following statements hold:

(A) $s_n = s_2$ for every $n > n_0$. Then there exists a sequence $x_n : r_1 < ||x_n|| < r_1 + 1/n$ such that $||y_n|| = ||u(x_n)|| \geq s_2 - 1/n$. Since $||x_n|| \to r_1$ it follows that $||u(x_n)|| \to s_1$. This is impossible.

(B) $s_1 < s_n < s_2$ for every $n > n'$. Since u is a surjection it follows that there exists a sequence $x_n : r_1 < ||x_n|| \leq r_1 + 1/n$ such that $||y_n|| = ||u(x_n)|| = s_n$. Since $||x_n|| \to r_1$ it follows that $||u(x_n)|| = s_n \to s_1$.

(C) There exist $n'' \in \mathbb{N}$ such that $s_n = s_1$ for every $n \geq n''$.

From (A), (B) and (C) we obtain $\lim_{n\to\infty} s_n = s_1$.

Let (B) hold. For every $n > n'$, let $\varepsilon_n = s_n - s_1$ such that $s_1 + 4\varepsilon_n < s_2$ and let φ_n be a C^2 real function defined on (s_1, s_2) by

$$\varphi_n(s) = \begin{cases} s_1 & \text{if } s_1 < s \leq s_1 + 2\varepsilon_n \\ h_n(s) & \text{if } s_1 + 2\varepsilon_n \leq s \leq s_1 + 4\varepsilon_n \\ s_2 + \frac{s_2 - s_1 - \varepsilon_n}{s_2 - s_1 - 4\varepsilon_n}(s - s_2) & \text{if } s_1 + 4\varepsilon_n \leq s < s_2 \end{cases}$$

where the function $h_n(t)$ satisfies the conditions: $h_n'(t) \geq 0$ and $h_n''(t) \geq 0$. An example of such a function is the function

$$h_n(s) = s_1 + \frac{s_2 - s_1 - \varepsilon_n}{s_2 - s_1 - 4\varepsilon_n} \int_{s_1+2\varepsilon_n}^{s} \left(\frac{\int_{s_1+2\varepsilon_n}^{x} (t - s_1 - 2\varepsilon_n)(s_1 + 4\varepsilon_n - t)\mathrm{d}t}{\int_{s_1+2\varepsilon_n}^{s_1+4\varepsilon_n} (t - s_1 - 2\varepsilon_n)(s_1 + 4\varepsilon_n - t)\mathrm{d}t} \right)^q \mathrm{d}x.$$

Here $q = q_n$ is chosen such that $h_n(s_1 + 4\varepsilon_n) = s_1 + \varepsilon_n$. This is possible because $\lim_{q \to +\infty} h_n(s_1 + 4\varepsilon_n) = s_1$ and

$$h_n(s_1 + 4\varepsilon_n)|_{q=1} = s_1 + \frac{s_2 - s_1 - \varepsilon_n}{s_2 - s_1 - 4\varepsilon_n} \frac{(s_1 + 4\varepsilon_n - s_1 - 2\varepsilon_n)}{2} > s_1 + \varepsilon_n.$$

It is obvious that

$$0 \leq \varphi_n'(s) \to 1 \quad \text{and} \quad 0 \leq \varphi_n''(s) \to 0 \quad \text{as } n \to \infty \tag{2.2}$$

for every $s \in (s_1, s_2)$. Let $\rho = ||u||$ and let ρ_n be the function defined on $\{x : r_1 < ||x|| < r_2\}$ by $\rho_n(x) = \varphi_n(\rho(x))$.

If (C) holds we can simply set $\rho_n(x) = \rho(x)$ and $\varphi_n(x) = x$. Then

$$\Delta\rho_n(x) = \varphi_n''(\rho(x))||\nabla\rho(x)||^2 + \varphi_n'(\rho(x))\Delta\rho(x).$$

By (2.2) it follows at once that

$$\Delta\rho_n(x) \to \Delta\rho(x) \quad \text{as } n \to \infty$$

for every $x \in A(r_1, r_2)$. Similarly, we obtain

$$\frac{\partial\rho_n}{\partial r}(x) \to \frac{\partial\rho}{\partial r}(x) \quad \text{as } n \to \infty$$

uniformly on $\{x : ||x|| = r\}$ for every $r \in (r_1, r_2)$. Applying Green's formula for ρ_n on $\{x : r_1 + 1/n \leq ||x|| \leq r\}$, we obtain

$$\int_{||x||=r} \frac{\partial\rho_n}{\partial r}\mathrm{d}S - \int_{||x||=r_1+1/n} \frac{\partial\rho_n}{\partial r}\mathrm{d}S = \int_{r_1+1/n \leq ||x|| \leq r} \Delta\rho_n dV.$$

Since the function ρ_n is constant in some neighborhood of the sphere $||x|| = r_1 + 1/n$, it follows that

$$\int_{||x||=r} \frac{\partial\rho_n}{\partial r}\mathrm{d}S = \int_{r_1+1/n \leq ||x|| \leq r} \Delta\rho_n dV.$$

Because of (1.4) and (2.2) it follows that the function $\Delta\rho_n$ is positive for every n. Hence, by applying Fatou's lemma, letting $n \to \infty$, we obtain

$$\int_{||x||=r} \frac{\partial\rho}{\partial r}\mathrm{d}S \geq \int_{r_1 \leq ||x|| \leq r} \Delta\rho dV.$$

Next, by applying (1.4), (1.11) and (1.12), we obtain

$$\int_{||x||=r}\frac{\partial\rho}{\partial r}\mathrm{d}S \geq \int_{r_1\leq||x||\leq r}\Delta\rho dV = \int_{r_1\leq|z|\leq r}\rho||S'||_2^2 dV$$
$$\geq s_1\int_{r_1\leq||x||\leq r}||S'||_2^2 dV \geq \begin{cases} 2\pi s_1\log\frac{r}{r_1} & \text{if } n=2,\\ 4\pi s_1(r-r_1) & \text{if } n=3.\end{cases}$$

It follows that

$$r^{n-1}\frac{\partial}{\partial r}\int_{||\zeta||=1}\rho\mathrm{d}S(\zeta) \geq \begin{cases} 2\pi s_1\log\frac{r}{r_1} & \text{if } n=2,\\ 4\pi s_1(r-r_1) & \text{if } n=3.\end{cases}$$

Dividing by r^{n-1} and integrating the previous inequality over $[r_1, r_2]$ with respect to r, we get

$$\int_{||\zeta||=1}\rho(r_2\zeta)\mathrm{d}S(\zeta) - \int_{||\zeta||=1}\rho(r_1\zeta)\mathrm{d}S(\zeta)$$
$$\geq \begin{cases} \pi s_1\log^2\frac{r_2}{r_1} & \text{if } n=2,\\ 4\pi s_1(\log\frac{r_2}{r_1} - r_1\frac{r_2-r_1}{r_2 r_1}) & \text{if } n=3.\end{cases}$$

Dividing by $2s_1$ if $n=2$ and by $4s_1$ if $n=3$, one gets

$$\frac{s_2}{s_1}-1 \geq \begin{cases} \frac{1}{2}\log^2\frac{r_2}{r_1} & \text{if } n=2,\\ \log\frac{r_2}{r_1} - \frac{r_2-r_1}{r_2} & \text{if } n=3.\end{cases}$$

Thus, the proof of the theorem has been completed. ∎

The following example justifies the Nitsche conjecture.

Example 2.2: Let $R = A(r,1)$ and let $R' = A(s,1)$. Then the function

$$f(z) = \frac{1-rs}{1-r^2}z + \frac{rs-r^2}{1-r^2}\frac{1}{\bar{z}} \tag{2.3}$$

is a harmonic diffeomorphism between R and R' if and only if

$$s \leq \frac{2r}{1+r^2}. \tag{2.4}$$

Moreover, under the same condition the function $u(z) = f(z^n)$ satisfies the conditions of Theorem 2.1 and maps the annulus $R = A(\sqrt[n]{r}, 1)$ onto the annulus $R' = A(s,1)$. From (2.4) it follows that

$$s \leq \frac{1}{1+1/2\log^2\sqrt[n]{r}}$$

(see the following remark).

Remark 2.3: (a) The inequality of Nitsche's conjecture is better than the inequalities obtained in this paper. It can be seen by the following. We have to prove the inequality

$$\frac{2r}{1+r^2} \leq \frac{1}{1+\frac{1}{2}\log^2 r}.$$

With $r = u^2$, the previous inequality is equivalent to

$$f(u) = \frac{1}{u} - u + 2\log u \geq 0.$$

Because of

$$f'(u) = -\Big(\frac{1-u}{u}\Big)^2 < 0 \quad \text{and} \quad f(1) = 0,$$

we deduce at once that $f(u) \geq 0$ and inequality is proved. Similarly, we can prove the inequality

$$\frac{2r}{1+r^2} \leq \frac{1}{r - \log r}.$$

Thus the Nitsche conjecture makes sense for the three-dimensional case, and it would be interesting to construct the corresponding canonical harmonic homeomorphism as in (2.3).

(b) On the other hand, the inequality obtained in this paper is better than Lyzzaik's inequality if the modulus of the domain is close to 0, and this fact easily follows from the inequalities obtained in [2] for the function $\mu(t)$, defined as the modulus of the Grötzsch's ring domain $R = \{z : |z| < 1\} \setminus [0, t]$.

(c) In [6], it has been observed that if there exists a harmonic diffeomorphism between two ring domains in the space, then the modulus of the co-domain cannot be small enough. The question arises which inequality holds for n-dimensional Euclid space, $n > 3$.

The following examples show that the converse inequality to inequality (2.1) does not hold. Moreover, there are harmonic diffeomorphisms between the annulus $A(1,2)$ and the arbitrarily large module annulus.

Example 2.4: Let f: $A(1,2) \to A(1,2^n)$ be the function defined by $f(z) = z^n$. Then f satisfies the condition of Theorem 2.1 and $m(A(1,2^n)) = n \cdot m(A(1,2))$.

Example 2.5: Let f: $A(1,2) \to A(0,1)$ be the function defined by

$$f(z) = \frac{2}{3}\Big(z - \frac{1}{\bar{z}}\Big).$$

Then f is a harmonic diffeomorphism.

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